Differential Geometry: Definitions and Pictures

Nic Ford

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We'll be discussing how these work and give some examples of how to use them to do geometry.

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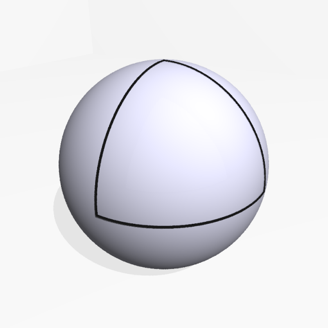

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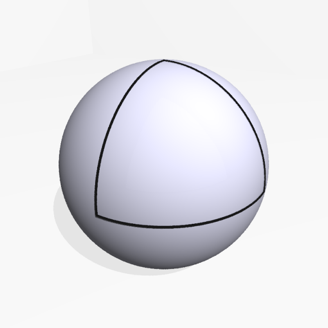

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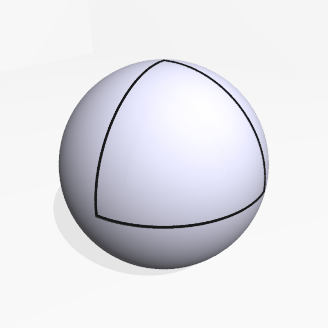

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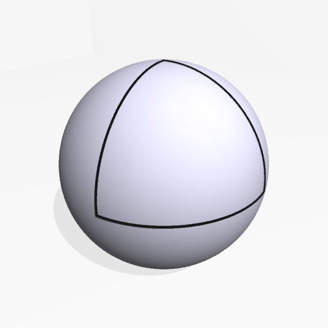

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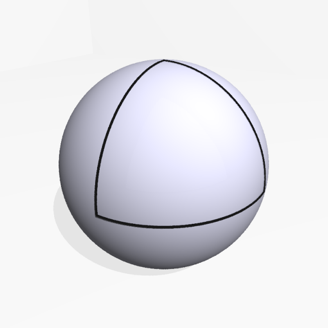

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Two natural coordinate systems on half of a sphere.

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...topology prevents you from using the same coordinates for the whole space.

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Even if you only care about $\mathbb{R}^n$, many aspects of calculus and geometry appear more natural with the general perspective in mind.

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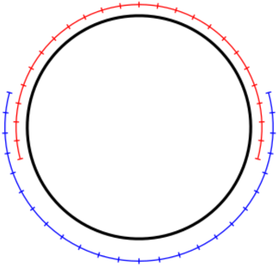

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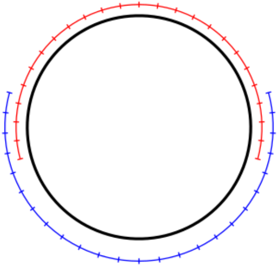
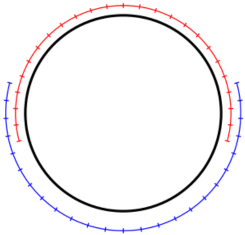


We require the coordinate-change functions from one chart to another to be smooth (i.e. infinitely differentiable).

Some examples



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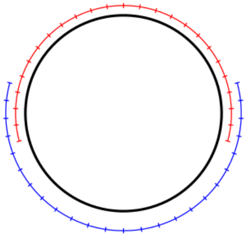

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Writing C for the circle, we can define the charts as:

$$r : \left(-\frac{\pi}{4}, \frac{5\pi}{4}\right) \to C \qquad b : \left(\frac{3\pi}{4}, \frac{9\pi}{4}\right) \to C$$

$$r(t) = (\cos t, \sin t) \qquad b(u) = (\cos u, \sin u)$$

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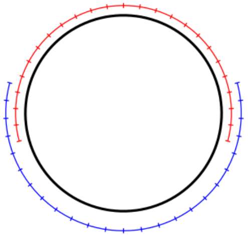

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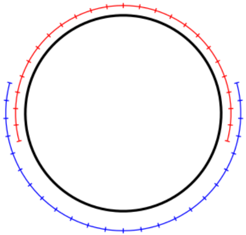

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The red-to-blue coordinate change function $b^{-1} \circ r$ is defined on $(-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$. It is the identity on the right half and $t \mapsto t + 2\pi$ on the left half.

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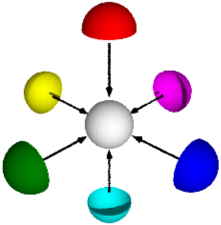

Six charts on a sphere.

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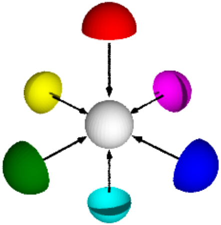

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Writing S for the sphere and B for the open unit ball in $\mathbb{R}^2$, two of these charts might be:

$$r : B \to S \qquad b : B \to S$$

$$r(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \qquad b(s, t) = (\sqrt{1 - s^2 - t^2}, s, t)$$

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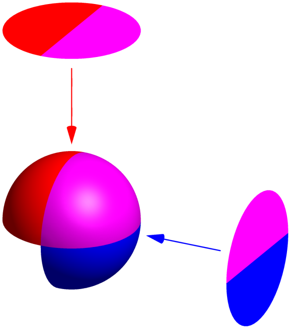

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$$(u, v) \mapsto (v, \sqrt{1 - u^2 - v^2}).$$

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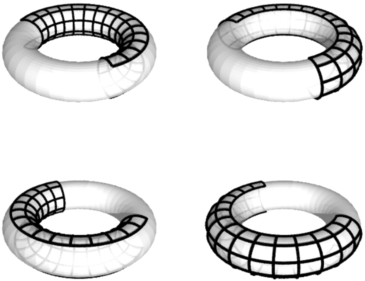

Four charts on a torus.

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The charts give many local coordinate systems that overlap with each other, but no particular one is special.

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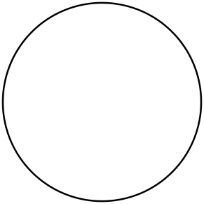
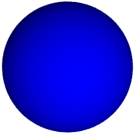
    on $\mathbb{R}^n$. The other m coordinates give the solution the structure of a manifold. (E.g., $z = 0$ defines a plane in $\mathbb{R}^3$, and we can use x and y to give in coordinates.)
  - The implicit function theorem says that, for *any* system of equations whose Jacobian matrix has full rank at p, we can find coordinates around p where the equations look like this.

Some examples

There are all smooth manifolds defined by one equation each:



$$x^2 + y^2 - 1 = 0$$



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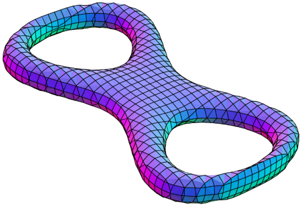
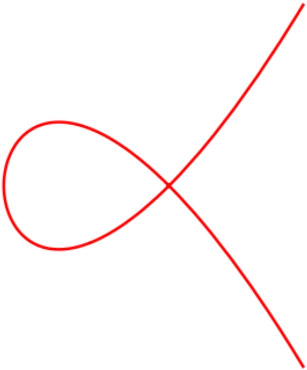


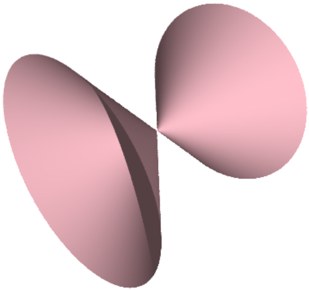
$$100((x(x-1)^2(x-2) + y^2)^2 + z^2) - 1 = 0$$

Some examples

These equations fail to define smooth manifolds because they violate the condition in the implicit function theorem at one point:



$y^2 - x^2 - x^3 = 0$



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Some examples

The set of all 3D rotations and reflections forms a three-dimensional manifold called $O(3)$.

Each rotation or reflection in $O(3)$ corresponds to a 3×3 matrix A for which $AA^T = 1$.

These conditions can be written as equations in the entries of the matrix, so $O(3)$ can be cut out by six equations in $\mathbb{R}^9$.

Building smooth manifolds

The method of defining manifolds as solutions to equations makes it even harder to think in terms of any particular coordinate systems!

Questions?

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We'll start by asking what it means for one of these to be smooth, and then we'll discuss how to differentiate them.

Smooth real-valued functions on manifolds

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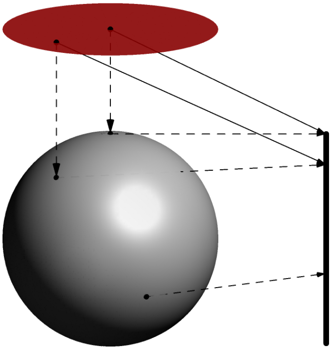

Example: the function on the sphere that takes each point to its z coordinate (as point in $\mathbb{R}^3$ with the usual embedding of the sphere).

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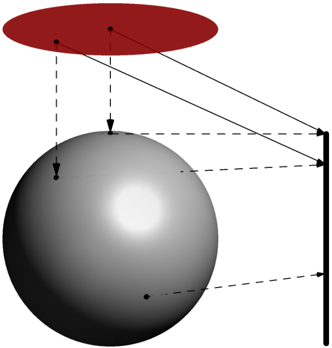

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Because we demanded that our coordinate-change functions have to be smooth, ϕ is smooth near some point p if and only if it's smooth on *any* chart containing p.

Smooth maps between manifolds

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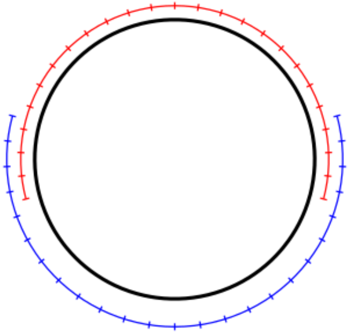
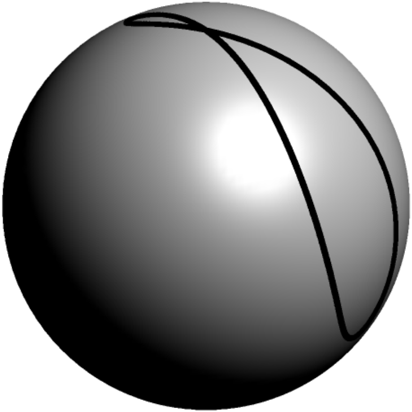
Our map is smooth if, for each $p \in M$, there's an M chart around p and an N chart around $f(p)$ on which the corresponding map between open balls in $\mathbb{R}^n$ is smooth (wherever it's defined).

Smooth maps between manifolds





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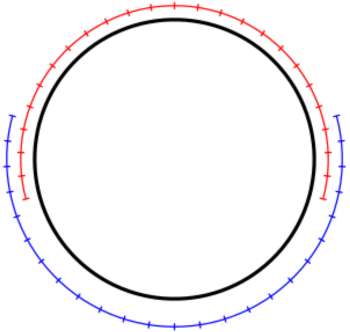

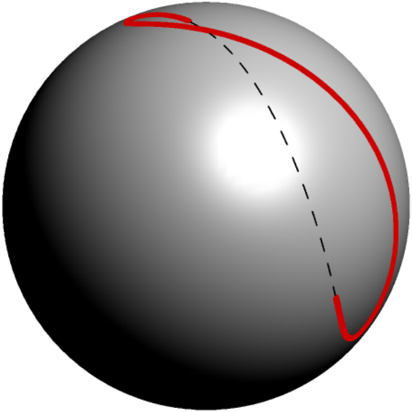

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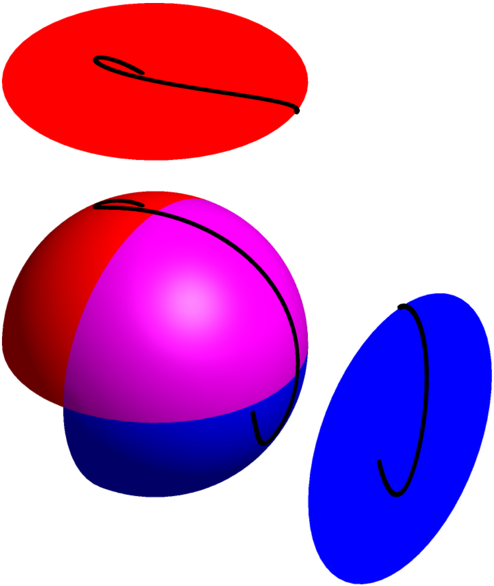
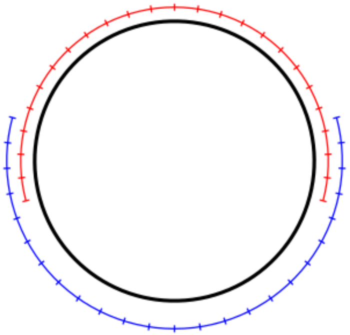
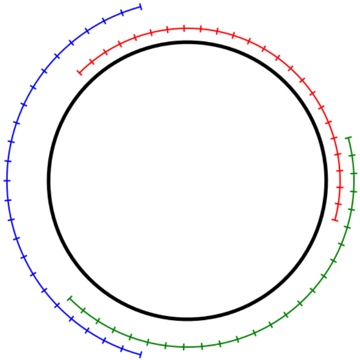


Smooth maps between manifolds

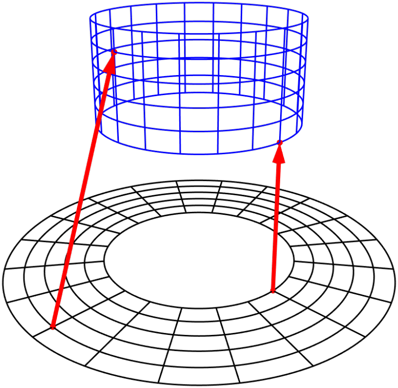
Again, if we get this on *any* collection of charts that cover the image of M, it follows for *every* chart we could check.

When are two manifolds “the same”?





Diffeomorphisms



Two manifolds are often thought of as "the same" if they are diffeomorphic: there's a one-to-one correspondence between two the points that's a smooth map in both directions.

Derivatives on manifolds

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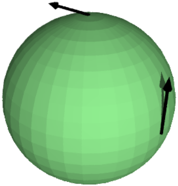

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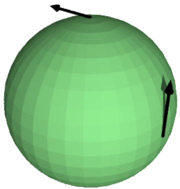

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So we should attach a *different* vector space to each point of the manifold, called the tangent space.

Defining tangent spaces

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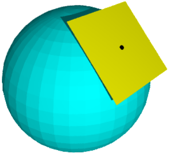

If the manifold is embedded in some $\mathbb{R}^n$, we can use the literal tangent space (or line, or plane). This picture is just a little misleading: tangent vectors should be thought of as extremely *close* to the point, not extending far away from it.

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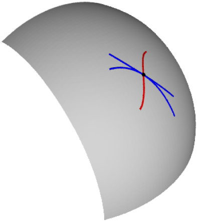

We can think of tangent vectors as being represented by tiny curves through the point — that is, smooth maps $\gamma : (-1, 1) \to M$ with $\gamma(0) = p$. We declare that two curves represent the same tangent vector as each other if they have the same *velocity*. It's harder to see how to add these, but it can be done.

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Tangent vectors and coordinates

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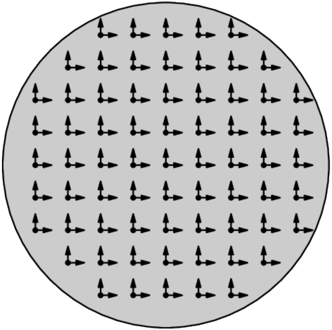

Picking a coordinate system on a chart gives rise to a collection of tangent vectors at every point in the chart called coordinate tangent vectors. At each point x, the coordinate tangent vectors for a basis for T_xM.

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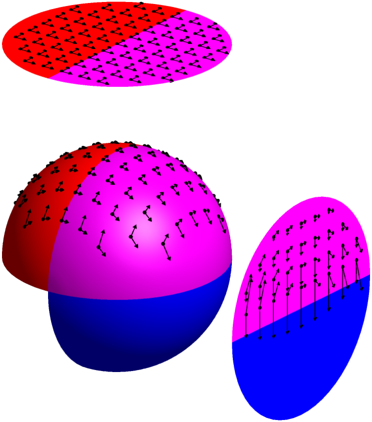

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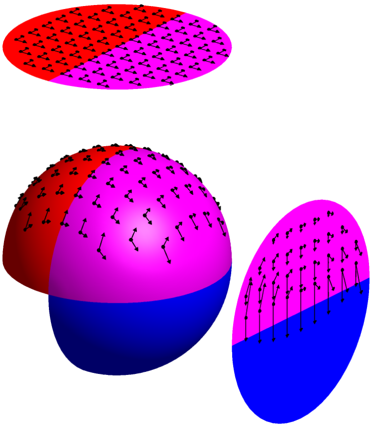

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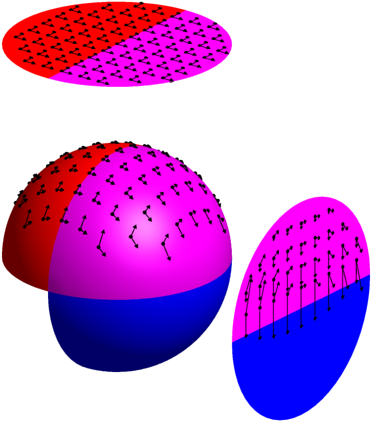

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The relationship between the coordinate vectors works out to:
$\hat{u} = (-\sqrt{1 - s^2 - t^2}/t)\hat{t}$; $\hat{v} = \hat{s} - (s/t)\hat{t}$.

Derivatives are linear maps between tangent spaces

Any smooth map $f : M \to N$ gives us a *linear* map $df_p : T_pM \to T_{f(p)}N$ on all of the tangent spaces called the pushforward or total derivative. (You might see the notation f_* too.) Think of it as "what f looks like if you zoom in very close to x."

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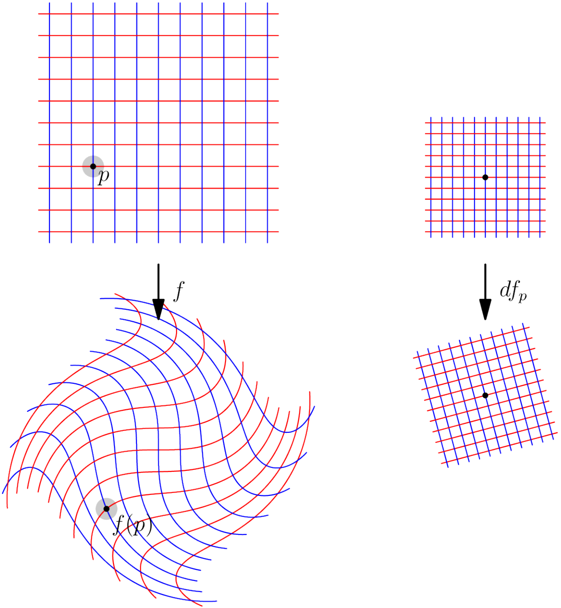

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When you pick charts around p and $f(p)$, the total derivative becomes the ordinary Jacobian matrix.

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

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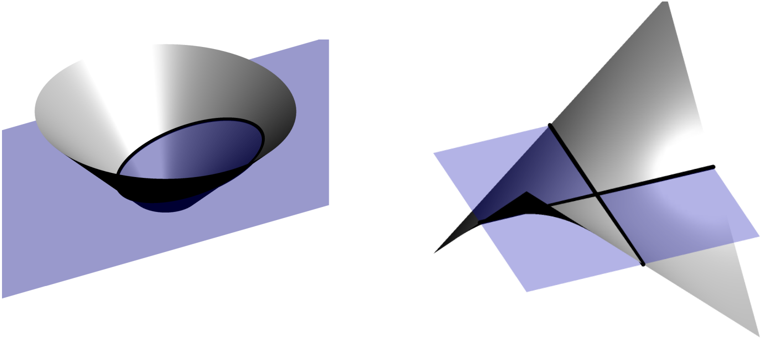

Example: if the intersection of two submanifolds is transverse — meaning their tangent spaces always intersect in the lowest dimension possible — the result is another manifold.

Extra miscellaneous tangent space facts

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From this perspective, the chain rule is just the fact that total derivatives respect composition of functions, that is, $d(g \circ f)_x = dg_{f(x)} \circ df_x$.

Questions?

Lengths and angles on manifolds

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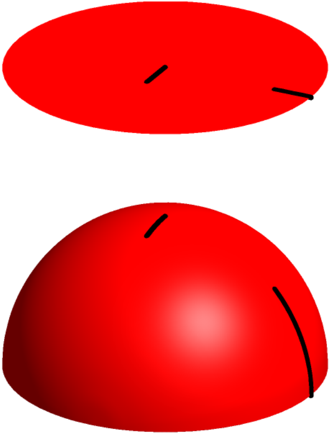

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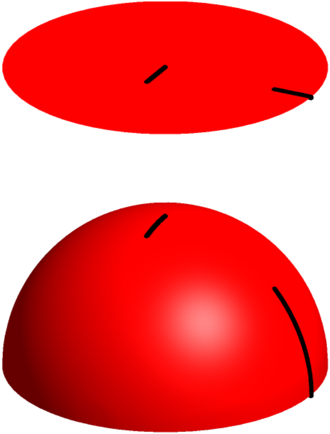
We need to put an extra piece of structure on our manifold to do this. The thing we'll use is called a Riemannian metric.



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- (bilinear) $\langle \alpha v + \beta v', w \rangle = \alpha \langle v, w \rangle + \beta \langle v', w \rangle$ and also on the other side.
- (positive definite) $\langle v, v \rangle \geq 0$, with equality only if $v = 0$.

Lengths and angles on manifolds

An inner product can be thought of as telling you about lengths and angles exactly like the ordinary dot product on $\mathbb{R}^n$:

$$\langle v, w \rangle = |v| \cdot |w| \cdot \cos\theta$$

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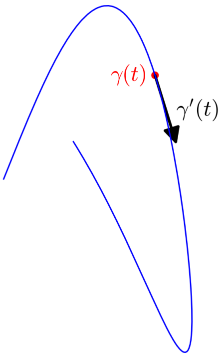
NB: The inner product is how we *define* lengths and angles! The right-hand sides of these two expressions should be taken as a way of interpreting a choice of inner product as an assignment of lengths and angles to vectors. But the symbols "$|\ |$" and "θ" have no meaning in isolation!

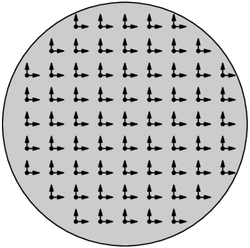
Lengths and angles on manifolds

Once we've picked a metric, we can, for example, compute lengths of curves. Given a smooth map $\gamma : [0, 1] \to M$, write $\gamma'(t)$ for the pushforward of the unit tangent vector at t. Then the length of γ is

$$\int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt.$$



Metrics in coordinates



Picking a coordinate chart gives us a basis of coordinate tangent vectors everywhere on the chart. If the coordinates are $x_1, \ldots, x_n$, write $\hat{e}_1, \ldots, \hat{e}_n$ for the coordinate tangent vectors. Once we've done this, we can write our metric as a matrix:

$$g = \begin{pmatrix} \langle \hat{e}_1, \hat{e}_1 \rangle & \langle \hat{e}_1, \hat{e}_2 \rangle & \cdots & \langle \hat{e}_1, \hat{e}_n \rangle \\ \langle \hat{e}_2, \hat{e}_1 \rangle & \langle \hat{e}_2, \hat{e}_2 \rangle & \cdots & \langle \hat{e}_2, \hat{e}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat{e}_n, \hat{e}_1 \rangle & \langle \hat{e}_n, \hat{e}_2 \rangle & \cdots & \langle \hat{e}_n, \hat{e}_n \rangle \end{pmatrix}.$$

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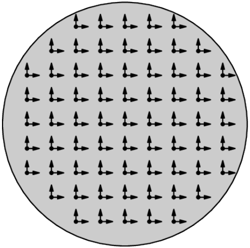

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As usual in linear algebra, we can then write $\langle v, w \rangle = v^T g w$, provided we expand v and w in the $\hat{e}_i$ basis.

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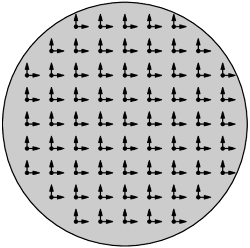

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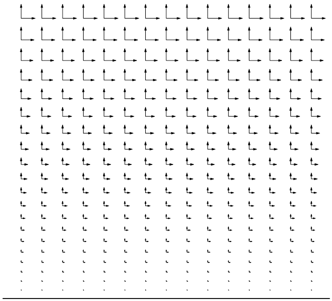
Note that the entries of the matrix will in general be *functions* of $x_1, \ldots, x_n$.

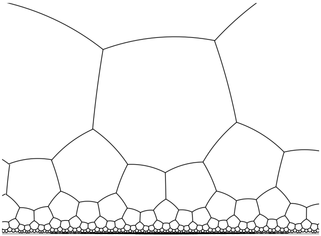
Example: the hyperbolic plane

The hyperbolic plane can be thought of as the upper-half plane with the inner product given by

$$\begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$$

in the standard basis of coordinate tangent vectors. (What does this mean about the *lengths* of the tangent vectors?)





(Credit: Wikipedia.)

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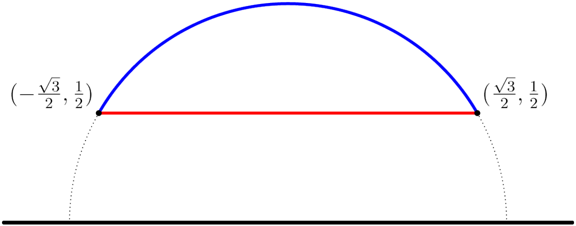

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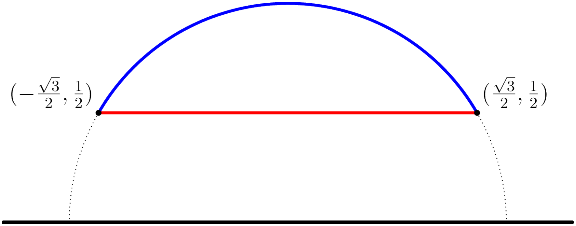

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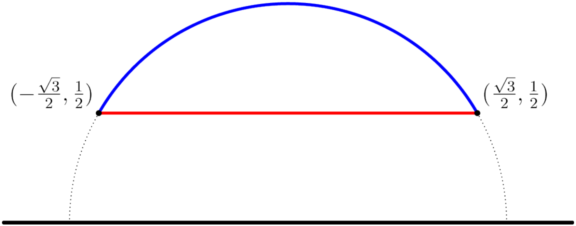

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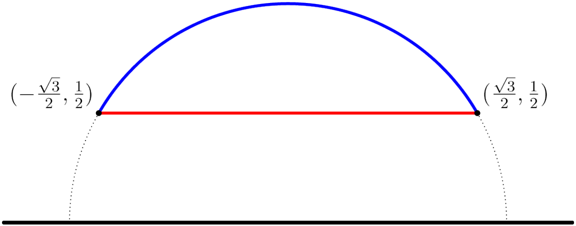

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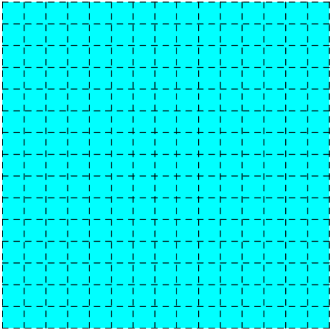
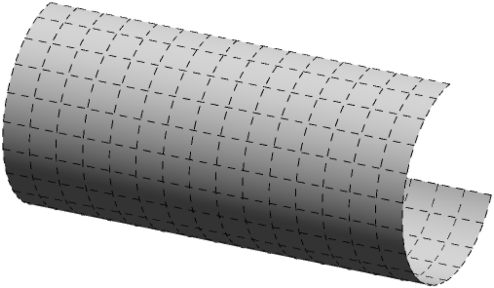
The blue curve can be given by $t \mapsto (\cos t, \sin t)$ for $\pi/6 \leq t \leq 5\pi/6$. The length of the tangent vector at time t is $1/\sin t$, so the total length is

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{\sin t}\, dt = \log\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) \approx 2.634.$$

Isometries

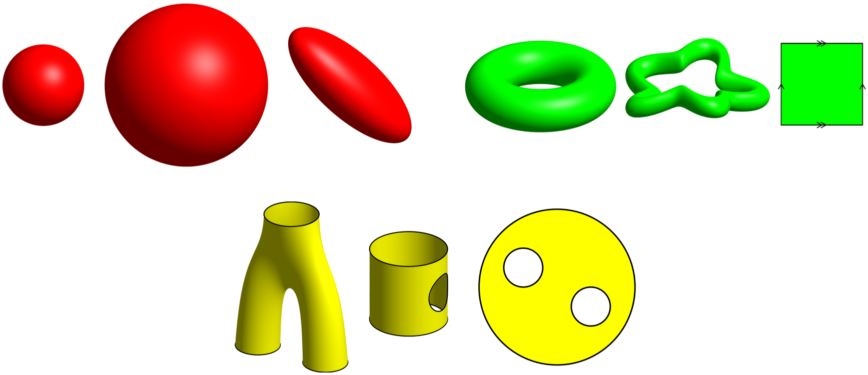
The presence of a metric introduces a new way for manifolds to be equivalent that's stronger than diffeomorphism: a diffeomorphism $f : M \to N$ is an isometry if it preserves the metric, that is, we always have $\langle v, w \rangle_{T_pM} = \langle df_p(v), df_p(w) \rangle_{T_{f(p)}N}$.





Isometries

Each group consists of manifolds that are diffeomorphic to each other but not isometric.



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If you have a coordinate system, you can use it to identify all the tangent spaces with $\mathbb{R}^n$, which gives you an obvious choice of metric. But most metrics don't arise in this way for *any* choice of coordinates! The ones that do are called flat. (The usual metric on the sphere, for example, isn't flat.)

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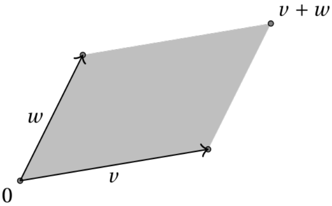

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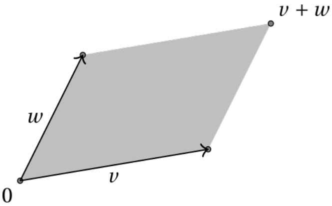

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In fact, knowing just the lengths is enough to determine the angles, provided that the lengths come from an inner product:

$$\langle v + w, v + w\rangle - \langle v - w, v - w\rangle = 4\langle v, w\rangle.$$

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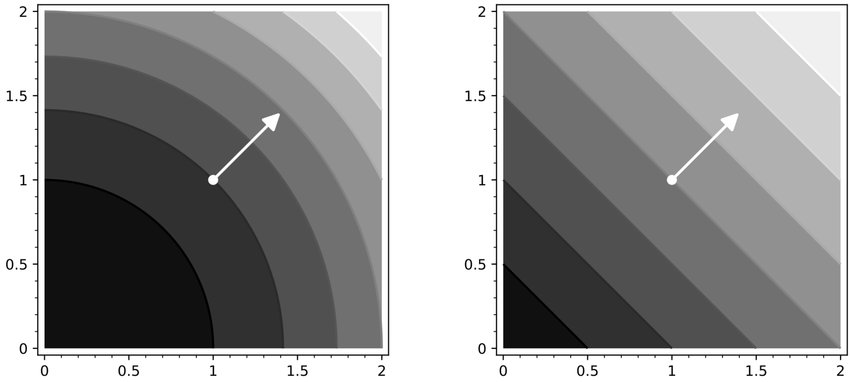

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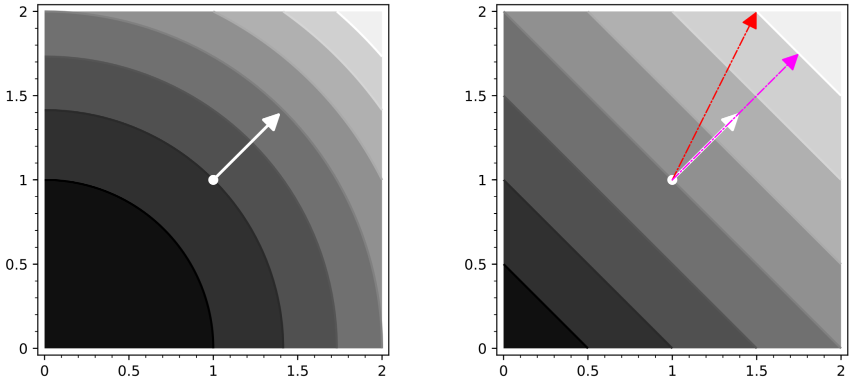

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- **General Relativity** by Robert Wald and **Gravitation** by Misner, Thorne, and Wheeler. Cover much of the same material motivated by physics (and then also cover the physics).