1 Motivating Measure

- We have an intuitive idea of length/area/volume, and it can be easy to see how it's supposed to work in a lot of cases. Questions can come up: how do we know that if we cut up a set in arbitrary ways, we won't get wrong answers about volume by adding the pieces? How do we assign measure to sets like the rationals between 0 and 1, where there are no whole segments?
- In fact, if you're not careful, you actually can run into trouble with this concept. We need a precise definition, called a *measure*.
- Motivate and enumerate:
  - Non-negative
  - Intervals have the right size
  - Countable additivity
  - Translation-invariance
- We can start by declaring that intervals have the right size. Handle 0 and ∞ correctly.
- We'll say *measure* for this formal thing, and *length* for the (obviously defined) size of a interval. The volume of an interval Q is written $|Q|$.
- Sets that can be written as unions of finitely many intervals should be assigned the obvious measure.
- Talk about open sets and topology. It seems like any set that's a countable union of intervals should be able to be assigned a measure, but we have problems with both existence and uniqueness of this representation. Talk about compactness and least upper bounds.

2 The Outer Measure

- From now on, we're one-dimensional. Other stuff might be on the homework.
- Let's be a little more careful. We won't try to get at the measure of a set by cutting it up exactly. Rather we'll take unions of intervals that approximate it on the outside.
- Define $m^*(S) = \inf_{S \subseteq \bigcup Q_i} \sum |Q_i|$, where the Q_i's are open intervals.
- Note the following properties:
  - Monotonicity. Clear.
  - Translation invariance. Clear.
  - Intervals have the right outer measure, whether open or closed. On the homework.
  - Countable subadditivity. Say $E = \bigcup E_j$. If any E_j has infinite measure, we're done, so say they don't. For any $\epsilon > 0$, cover E_j by intervals Q_{jk} of total length $m^*(E_j) + \epsilon/2^j$. Then we obtain that $m^*(E) \leq \sum m^*(E_j) + \epsilon$.

- $m^*(S) = \inf m^*(U)$, where U runs over the open sets containing S. One way is clear. For the other, cover S with intervals of total volume $m^*(S)+\epsilon$. We can surround each interval with an open interval $\epsilon/2^j$ bigger, and there's our open set.
- If E_1 and E_2 are a positive distance apart, their outer measures add. On the homework.
- If E can be decomposed into a countable disjoint union of intervals, the measure is what you'd expect. You can shrink each interval by $\epsilon/2^j$, thereby shrinking the total length by just ϵ, but then everything is a positive distance apart. So the outer measures add, giving that $m^*E \geq (\text{expected}) - \epsilon$.

- We can't, though, assert the general disjoint additivity thing for outer measure. Still, it seems like this is the definition we want. The solution is to only assign measures to certain sets.

3 Measurability

- How can we make disjoint additivity work out? Given two sets E_1 and E_2, we want to be able to slice off the part of the union that belongs to E_1 without changing the total measure. That is, writing $A = E_1 \cup E_2$, we want $m^*(A) = m^*(E_1) + m^*(A \setminus E_1)$. Since E_2 could be anything at all, all we know about A is that it contains E_1. Things work out nicer if we don't even insist on that, so that we say E is *measurable* if, for any A at all, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$.
- Note that $\leq$ is automatic, so we only need to check $\geq$.
- We have the following properties:
  - Complements of measurable sets are measurable. Clear by definition.
  - If $m^*(E) = 0$, E is measurable. On the homework.
  - Lemma: if E_1 and E_2 are measurable, so is $E_1 \cup E_2$. Since E_2 is measureable,
  $$m^*(A \setminus E_1) = m^*((A \setminus E_1) \cap E_2) + m^*((A \setminus E_1) \setminus E_2),$$
  and since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \setminus E_1)$, we know that
  $$m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \setminus E_1).$$
  So we get that
  $$\begin{aligned} m^*(A \cap (E_1 \cup E_2) + m^*(A \setminus (E_1 \cup E_2)) &\leq m^*(A \cap E_1) + m^*(A \cap E_2 \setminus E_1) + m^*(A \cap (E_1 \cup E_2)) \\ &= m^*(A \cap E_1) + m^*(A \setminus E_1) \\ &= m^*A. \end{aligned}$$
  - Therefore we get intersections, differences, etc.
  - Lemma: if $E_1, \ldots, E_n$ are disjoint and measurable, then $m^*\left(A \cap \bigcup E_j\right) = \sum m^*\left(A \cap E_j\right)$. Clear for $n = 1$, so say it works for $n-1$. Then, since $A \cap \bigcup E_i \cap E_n = A \cap E_n$ and $A \cap \bigcup E_i \setminus E_n = A \cap \bigcup^{n-1} E_i$, we get by the measurability of E_n that
  $$\begin{aligned} m^*\left(A \cap \bigcup E_i\right) &= m^*(A \cap E_n) + m^*\left(A \cap \bigcup^{n-1} E_i\right) \\ &= m^*(A \cap E_n) + \sum^{n-1} m^*(A \cap E_i) \end{aligned}$$

by induction.

- Countable unions of measurable sets are measurable. We can assume without loss of generality that the sets are disjoint. So say $F_n = \bigcup^n E_i$. We know it's measurable and contained in the big union E, so its complement contains E^c. So

$$m^*A = m^*(A \cap F_n) + m^*(A \setminus F_n) \geq m^*(A \cap F_n) + m^*(A \setminus E),$$

and so we learn from the previous lemma that $m^*(A \cap F_n) = \sum^n m^*(A \cap E_i)$, and so, dropping this in the previous line, we get that $m^*A \geq \sum^n m^*(A \cap E_i) + m^*(A \setminus E)$. But then we can let n go to infinity, letting us conclude that

$$m^*A \geq \sum^{\infty} m^*(A \cap E_i) + m^*(A \setminus E) \geq m^*(A \cap E) + m^*(A \setminus E)$$

by countable subadditivity.

- We have the disjoint additivity business for measurable sets. We have this for finitely many things already. So say we have a disjoint sequence of measurable sets. Then we know that $m\left(\bigcup^{\infty} E_i\right) \geq m\left(\bigcup^n E_i\right) = \sum^n m E_i$. But then we can let n go to infinity on the right and conclude that $m\left(\bigcup^{\infty} E_i\right) \geq \sum^{\infty} m E_i$.
- The open interval is measurable. This is on the homework.

- This gives us a whole bunch of measurable sets. We can measure anything which can be written by taking open sets, and taking unions, complements, and intersections countably many times in any order. These are called *Borel sets*.
- In fact, all measurable sets are very close to being Borel sets, in a sense that you'll explore more fully on the homework.

4 The Vitali Set

- We can put an equivalence relation on the reals by saying two numbers are equivalent if they differ by a rational.
- Pick a representative from each equivalence class within $[0,1]$ and call the resulting set V.
- Let $q_1, q_2, q_3, \ldots$ run through the rationals in $[-1,1]$. Then the sets $V + q_i$ are all disjoint: if $x \in V + q_i \cap V + q_j$, then $v + q_i = v' + q_j$, so v and v' differ by a rational, and therefore they must be equal by the construction of V.
- But $[0,1] \subseteq \bigcup_i (V + q + i) \subseteq [-1,2]$, so if V had a measure, then by disjoint additivity and translation-invariance, $1 \leq \sum_{i=1}^{\infty} m V \leq 3$, which can't happen.